Intro Video: Section 3.1 Derivatives of Polynomials and Exponential Functions

Math F251X: Calculus I

Polynomials: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

a_i are constants, called coefficients

the **degree** is the highest power

Example: $f(x) = 3x^2 - 5x + 6$ is a degree 2 (quadratic) polynomial

Exponential function: $f(x) = a^x$ where a is a real #, $a \neq 0$.

New Notation for differentiation!

$\frac{d}{dx}$(some function) $\rightarrow$ returns the derivative!

write $\boxed{\frac{df}{dx} \text{ or } f'(x)}$

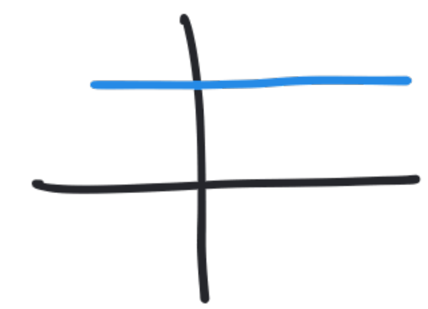
to mean the derivative of f with respect to x

FACTS:

① $\frac{d}{dx}(\text{constant}) = 0$

$f(x) = c$:

$$\lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

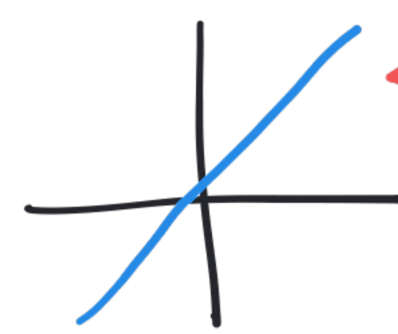


← Slope = 0

② $\frac{d}{dx}(x) = 1$

$f(x) = x$

$$\lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$



← Slope = 1

③ $\frac{d}{dx}(x^2) = 2x$

$f(x) = x^2$:

$$\lim_{h \to 0} \frac{(x-h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}} = \lim_{h \to 0} 2x + h = 2x$$

What is $\frac{d}{dx}(x^n)$?

We can prove a result when n is a positive integer.

<u>Fact</u>: $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \dots + x^i a^{n-1-i} + x a^{n-2} + a^{n-1})$

If $f(x) = x^n$, then:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = \lim_{x \to a} \frac{x^n - a^n}{x-a}$$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1})}{x-a}$$

$$= \lim_{x \to a} x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1}$$

$$= a^{n-1} + a^{n-2}a + \dots + a a^{n-2} + a^{n-1}$$

$$= n \cdot a^{n-1}$$

FACT: $\frac{d}{dx}(x^n) = n x^{n-1}$

POWER RULE: If n is any constant,

$$\frac{d}{dx}\left(x^{n}\right) = n\,x^{n-1}$$

Examples:

$$\frac{d}{dx}\left(x^{4}\right) = 4x^{3}$$

$$\frac{d}{dx}\left(x^{3/2}\right) = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{3/2-2/2} = \frac{3}{2}x^{1/2}$$

$$\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}\,x^{\sqrt{2}-1}$$

Arithmetic of derivatives, part 1:

1. $$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}\left(f(x)\right) + \frac{d}{dx}\left(g(x)\right)$$

Example: $$\frac{d}{dx}\left(x^2+\frac{1}{x}\right) = \frac{d}{dx}\left(x^2+x^{-1}\right)$$
$$= \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(x^{-1}\right)$$
$$= 2x + (-1)x^{-2}$$

2. $$\frac{d}{dx}\left(f(x)-g(x)\right) = \frac{d}{dx}\left(f(x)\right) - \frac{d}{dx}\left(g(x)\right)$$

3. $$\frac{d}{dx}\left(c\,f(x)\right) = c\cdot\frac{d}{dx}\left(f(x)\right)$$

Example: Compute $\frac{d}{dx}\left(3x^2 + 2x - \sqrt{5x}\right)$

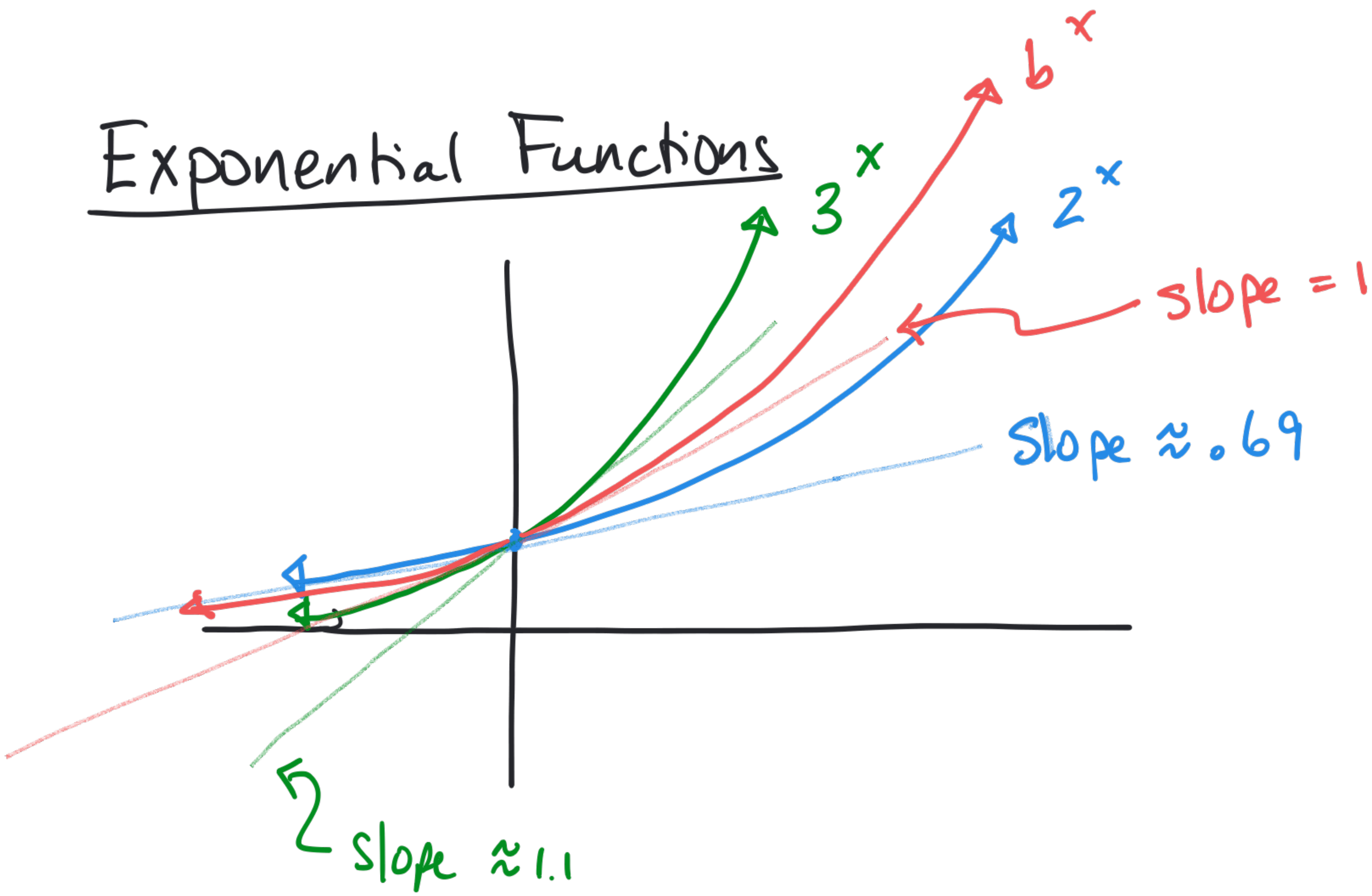
$$\frac{d}{dx}\left(3x^2+2x-\sqrt{5x}\right) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}\left(\sqrt{5}\cdot\sqrt{x}\right)$$

$$= 3\,\frac{d}{dx}(x^2) + 2\,\frac{d}{dx}(x) + \sqrt{5}\cdot\frac{d}{dx}\left(x^{1/2}\right)$$

$$= 3(2x) + 2(1) + \sqrt{5}\left(\tfrac{1}{2}x^{-1/2}\right)$$

STOP HERE! You do not need to simplify (unless you need the derivative for another purpose.)

Exponential Functions



FACT

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(b^x) = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} \frac{b^x b^h - b^x}{h} = \lim_{h\to 0} \frac{b^x(b^h - 1)}{h}$$

$$= b^x \boxed{\lim_{h\to 0} \frac{b^h - 1}{h}}.$$

We define e to be the number so that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$